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Existence of the Unique kT-periodic Solution for One Class of Nonlinear Systems

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The system of ordinary differential equations with the discontinuous hysteresis nonlinearity and an external continuous biharmonic influence is considered. Sufficient conditions are obtained for existence of periodic solutions with given properties, in particular, with the period equal and multiple to the period of the external influence. We use an approach to the choice of feedback coefficients based on a nonsingular transformation of the initial system to a special canonical form. The approach allows to find analytically the switching instants and points of the image point of the required solution.

Keywords: forced periodic oscillations, nonsingular transformations, control systems, discontinuous hysteresis nonlinearity, switching instants and points.

Introduction

In this paper we study problems of existence of forced oscillatory solutions in control systems containing a discontinuous hysteresis nonlinearity and the possibility to choose coefficients of the system so that there are oscillatory modes with the given properties in it and a certain configuration in the phase space. Systems of the considered class can be used studying the processes in electric drive systems constructed with semi-conductor diodes and intended for regulation of the rotation frequency of a rotor with an asynchronous electric motor, or studying processes in electric chains of control systems using a nonideal relay and elements from ferromagnetic materials.

In recent years solvability of problems of control systems with discontinuous nonlinearities has been established by a variational method [1–3], method of regularization and theory of a topological degree for multivalued compact vector fields [4].

This paper is based on general results related to dynamics of higher order systems with hysteresis nonlinearities stated by V.I. Zubov in [5], and on methods of theory of canonical transformations. We use the method of sections for the system parameter space suggested by R.A. Nelepin [6] in the case of autonomous systems and allowing to reduce a system of higher order to systems of the 1st and 2nd orders. The latter ones are already well studied and can be investigated by exact analytical methods, namely, by the method of images and the fixed point method.

In [7], we consider a nonperiodic external influence such that its amplitude changes with the time. In this paper results of both V.I. Zubov and R.A. Nelepin are developed for systems at issue in the case of complicated biharmonic external influence.

Problems of the existence of periodic solutions with the given properties at biharmonic external influence were studied by the author earlier (see, e.g., the bibliography in [8]). In [8], the

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theorem of existence of superharmonious oscillations (that also can appear in the system at a periodic external influence) is proved. Comparing with [8], in this paper the emphasis is made on the analysis of the coefficient space of the initial system and finding sufficient conditions for the existence of subharmonic solutions. Unlike the previous investigations, in the present paper, for the first time, all sufficient parametrical conditions are obtained and formulated in the form of the theorem with the proof.

1. Problem Statement

A system of ordinary differential equations of the form

$$\dot{Y} = AY + BF(\sigma) + Kf(t), \quad \sigma = (C, Y) \quad (1)$$

is considered in n -dimensional real Euclidean space E^n . Here A is a matrix, vectors B, K, C are real and constant, Y is a state vector of the system.

The function $F(\sigma)$ describes hysteresis nonlinearity of a non-ideal relay type with threshold numbers ℓ_1, ℓ_2 and output numbers m_1, m_2 . To be specific, put $\ell_1 < \ell_2$ and $m_1 < m_2$. The function $F(\sigma(t))$ with a continuous input $\sigma(t)$ is defined for all $t \geq 0$ in a class of piecewise-continuous functions and it is given as follows. When $\sigma(t) \leq \ell_1$, the equality $F(\sigma) = m_1$ holds and when $\sigma(t) \geq \ell_2$, the equality $F(\sigma) = m_2$ holds. If $\ell_1 < \sigma(t) < \ell_2$ for all $t_1 < t \leq t_2$, then $F(\sigma(t_1)) = F(\sigma(t_2))$ [9]. The latter case means that at the instant t_1 the function $\sigma(t)$ can take only one value of the function $F(\sigma)$ because $\sigma(t_1) \leq \ell_1$ or $\sigma(t_1) \geq \ell_2$ and in a non-single-valued zone of the nonlinearity for all $t_1 < t \leq t_2$ the function $\sigma(t)$ takes two values of the function $F(\sigma)$ but the value of the function $F(\sigma)$ should be chosen being equal to $F(\sigma(t_1))$.

The values of the function $F(\sigma(t))$ for the continuous input $\sigma(t)$ and all $t \geq 0$ are completely defined as follows. The function $F(\sigma(t))$ takes the constant value on the closed interval $[t_1, t_2]$ if either $F(\sigma(t_1)) = m_1$ and $\sigma(t) < \ell_2$ for all $t \in [t_1, t_2]$ or $F(\sigma(t_1)) = m_2$ and $\sigma(t) > \ell_1$ for all $t \in [t_1, t_2]$. This rule is known as the principle of the absence of unnecessary switchings.

The function $F(\sigma)$ describes a relay loop, or, the same, the counterclockwise run on the plane $(\sigma, F(\sigma))$. The graph of this function is shown in figure 1.

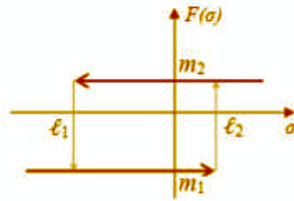


Fig. 1. The graph of the hysteresis function

Note that the function $F(\sigma) = F((C, Y))$ describes, for example, a space delay with respect to the phase coordinates of control mechanisms [6]. A real n -dimensional vector C defines a feedback in the system.

The function $f(t)$ describes a continuous biharmonic external influence on the system of the following form:

$$f(t) = f_0 + f_1 \sin(\omega t + \varphi_1) + f_2 \sin(2\omega t + \varphi_2), \quad (2)$$

where $f_0, f_1, f_2, \varphi_1, \varphi_2, \omega$ are nonzero and real constants.

Definition. A *switching point* of a solution to the system (1) in the phase space is such a state of the system that the function $\sigma(t)$ takes one of its threshold numbers and the control function changes an output number, i.e. the switch over occurs in relay.

Consider the problem of the existence of continuous periodic solutions with two switching points fixed in the phase space and the period multiple to the period of the function describing external influence.

The solution of this problem consists in proving the solvability of an auxiliary system of transcendental equations in an analytical form with respect to the switching instants and the instants of the image point of the periodic solution in the phase space. The second switching instant coincides with the given period of the desired solution of the system. The system of transcendental equations and its solvability conditions are given in [10].

2. Results

First we consider a nonsingular transformation reducing the initial system to the canonical form.

Let eigenvalues λ_i ($i = \overline{1, n}$) of the matrix A be real, prime, and nonzero. Besides, let us assume that the system (1) is completely controllable with respect to the input $F(\sigma)$, i.e. the inequality $\det \|B, AB, A^2B, \dots, A^{n-1}B\| \neq 0$ holds. Then the system (1) can be reduced to the canonical form by a nonsingular transformation $Y = SX$:

$$\dot{X} = A_0X + B_0F(\sigma) + K_0f(t), \quad \sigma = (\Gamma, X), \quad (3)$$

where

$$A_0 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad B_0 = S^{-1}B = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix}, \quad K_0 = S^{-1}K, \quad \Gamma = \begin{pmatrix} \gamma_1 \\ \dots \\ \gamma_n \end{pmatrix}.$$

The coefficients γ_i ($i = \overline{1, n}$) are calculated by the formula:

$$\gamma_i = \frac{-1}{D'(\lambda_i)} \sum_{k=1}^n c_k N_k(\lambda_i), \quad (4)$$

where $D'(\lambda_i) = \left. \frac{dD(p)}{dp} \right|_{p=\lambda_i}$, $N_k(\lambda_i) = \sum_{i=1}^n b_i D_{ik}(\lambda_i)$.

Here $D_{ik}(p)$ is a co-factor of the corresponding element (this element is situated on the crossing of the line i and the column k) of the determinant $D(p)$; λ_i are the roots of the algebraic equation $D(p) = \det [a_{k\alpha} - \delta_{k\alpha}p] = 0$, $a_{k\alpha}$ are the elements of the matrix A , $\delta_{k\alpha}$ is Kroneker's symbol. The transformation matrix S has the form

$$S = - \begin{pmatrix} \frac{N_1(\lambda_1)}{D'(\lambda_1)} & \dots & \frac{N_1(\lambda_n)}{D'(\lambda_n)} \\ \vdots & \ddots & \vdots \\ \frac{N_n(\lambda_1)}{D'(\lambda_1)} & \dots & \frac{N_n(\lambda_n)}{D'(\lambda_n)} \end{pmatrix}. \quad (5)$$

The main result of this paper is the following theorem.

Theorem 1. *Let the following conditions hold:*

- 1) *the external influence $f(t)$ of the system (1) is a T -periodic function of the form (2), where $T = 2\pi/\omega$;*
 - 2) *the inequalities $-(C, A^{-1}Bm_2) < \ell_1$, $-(C, A^{-1}Bm_1) > \ell_2$ hold;*
 - 3) *the system (1) is reduced to the canonical form (3) by the nonsingular transformation $Y = SX$, where the form of the matrix S is determined by the formula (5), if the system is completely controllable with respect to the input $F(\sigma)$ and the eigenvalues of the matrix A are prime, nonzero, and real;*
 - 4) $\sum_{k=1}^n c_k N_k(\lambda_j) = 0$ ($j = \overline{1, n}$, $j \neq s$), $\sum_{k=1}^n c_k N_k(\lambda_s) \neq 0$, *where c_k are the components of the feedback vector C , N_k is the determinant of the matrix A in which the elements of the column with number k are replaced by components of the vector B , s is some index running the values from 1 to n ;*
 - 5) (t_1, kT, X^1, X^2) *is a solution to the system of the transcendental equations which was constructed under the assumption that there exists at least one periodic solution with two switching points and that the system's parameters satisfy to its solvability conditions for given $k \in \mathbf{N}$, where t_1 and kT are the first and the second switching instants respectively and X^1, X^2 are switching points of the image point of the solution to the canonical system.*
- Then there exists a unique kT -periodic solution to the system (1) with two switching points Y^1, Y^2 belonging to the hyperplanes of the form $(C, Y) = \ell_i$ ($i = 1, 2$), where $Y^1 = SX^1, Y^2 = SX^2$.*

Proof. In the case of a T -periodic external influence, a nonlinear system may admit harmonic and subharmonic oscillations with the period kT , where $k \in \mathbf{N}$.

Let us look for solutions to the system (1) in the class of continuous, bounded functions with two switching points lying in the hyperplanes of the form $(C, Y) = \ell_i$ ($i = 1, 2$) and the returning time being equal to $T_B = kT$, where $k \in \mathbf{N}$, for the image point of the solution at each switching point in the phase space.

To provide the continuity of the solution $Y(t)$, let the switching points (in which the right hand side of the system (1) changes) coincide with "join" points of the trajectories constructed by virtue of various right parts of the initial system, i.e. by virtue of the linear systems of the following form:

$$\dot{Y} = AY + Bm_1 + Kf(t), \quad \dot{Y} = AY + Bm_2 + Kf(t). \quad (6)$$

The variable t_1 is defined as the first switching instant of the image point of system's solution and hence the inequality $0 < t_1 < kT$ holds for $k \in \mathbf{N}$.

Let us define the sequence of motions for the image point of the desired solution from one hyperplane to other one. Let the image point of the desired periodic solution to the system (1) begin its motion at the point Y^1 on the hyperplane $\sigma = \ell_1$ at the instant $t_0 = 0$ and reach the point Y^2 on the hyperplane $\sigma = \ell_2$ at the instant t_1 by virtue of (6) provided that $m_i = m_1$. Then it returns to the point Y^1 on the hyperplane $\sigma = \ell_1$ at the instant T_B by virtue of (6) provided that $m_i = m_2$.

Condition 2) of Theorem 1 is sufficient for the image point of the considered solution to reach the hyperplanes as this means that the system (1) has $\dot{Y} = 0$ at the points $Y_i = -A^{-1}Bm_i$ ($i = 1, 2$) whenever $f(t) \equiv 0$. These virtual points of the stability in the phase space of the system lie out of the non-single-valued zone of the function $F(\sigma)$.

Taking into account the necessary conditions for the existence of the periodic solution with the given properties, let us construct the auxiliary system of transcendental equations with respect to the switching points Y^1, Y^2 , the first switching instant t_1 , and the time for returning (i.e. the period) $T_B = kT$. Using Cauchy's form for the solution, we have

$$\ell_1 = (C, Y^1), \quad \ell_2 = (C, Y^2), \quad (7)$$

where

$$\begin{aligned} Y^2 &= e^{At_1} Y^1 + \int_0^{t_1} e^{A(t_1-\tau)} (Bm_1 + Kf(\tau)) d\tau, \\ Y^1 &= e^{A(T_B-t_1)} Y^2 + \int_{t_1}^{T_B} e^{A(T_B-\tau)} (Bm_2 + Kf(\tau)) d\tau. \end{aligned}$$

In the general case, this system can not be solved in an analytical form. To simplify the system of the transcendental equations, let us use the canonical transformation of the initial system in accordance with Condition 3) of Theorem 1.

Further, following [6], we assume that $(n-1)$ roots of the equation $D(p) = 0$ coincide with $(n-1)$ roots of the equation $\sum_{k=1}^n c_k N_k(p) = 0$. Then, $(n-1)$ values of γ_i determined by the formula (4) vanish (see Condition 4) of Theorem 1).

The canonical system of the n -th order splits into lower-order systems that can be integrated successively if the coefficients γ_i ($i = \overline{1, n}$) are chosen according to Condition 4). The function $\sigma(t) = (\Gamma, X(t))$ is defined from the first order system

$$\sigma(t) = \gamma_s x_s, \quad \dot{x}_s = \lambda_s x_s + F(\sigma) + k_s^0 f(t),$$

the remaining variables x_i ($i \neq s$) are defined from non-homogeneous linear equations of the first order

$$\dot{x}_i = \lambda_i x_i + F(\sigma) + k_i^0 f(t), \quad i \neq s.$$

The differential equation

$$\dot{\sigma}(t) = \lambda_s \sigma(t) + \gamma_s (F(\sigma(t)) + k_s^0 f(t))$$

with respect to the function $\sigma(t)$ has a solution in the general form

$$\sigma(t) = \sigma_0 e^{\lambda_s(t-t_0)} + \gamma_s e^{\lambda_s t} \left(m_i \int_{t_0}^t e^{-\lambda_s \tau} d\tau + k_s^0 \int_{t_0}^t e^{-\lambda_s \tau} f(\tau) d\tau \right)$$

with the initial and boundary conditions according to the ordered sequence of the motions for the image point of the solution from one switching hyperplane to other hyperplane

$$\ell_1 = \sigma(\ell_1, 0, m_1, 0), \quad \ell_2 = \sigma(\ell_1, 0, m_1, t_1), \quad \ell_1 = \sigma(\ell_2, t_1, m_2, T_B).$$

Thus the system of the transcendental equations (7) is simplified and transformed into the system of two equations concerning only with the first switching instant t_1 , the time for returning T_B

$$\begin{aligned} \ell_2 &= \left(\ell_1 + \frac{\gamma_s m_1}{\lambda_s} \right) e^{\lambda_s t_1} - \frac{\gamma_s m_1}{\lambda_s} + \gamma_s k_s^0 \int_0^{t_1} e^{\lambda_s(t_1-\tau)} f(\tau) d\tau, \\ \ell_1 &= \left(\ell_2 + \frac{\gamma_s m_2}{\lambda_s} \right) e^{\lambda_s(T_B-t_1)} - \frac{\gamma_s m_2}{\lambda_s} + \gamma_s k_s^0 \int_{t_1}^{T_B} e^{\lambda_s(T_B-\tau)} f(\tau) d\tau \end{aligned} \quad (8)$$

and formulae for finding the switching points X^1 , X^2 of the canonical system (3)

$$\begin{aligned} X^1 &= (E - e^{A_0 T_B})^{-1} \left(\int_{t_1}^{T_B} e^{A_0(T_B-\tau)} (B_0 m_2 + K_0 f(\tau)) d\tau + \right. \\ &\quad \left. + \int_0^{t_1} e^{A_0(T_B-\tau)} (B_0 m_1 + K_0 f(\tau)) d\tau \right), \\ X^2 &= (E - e^{A_0 T_B})^{-1} \left(\int_0^{t_1} e^{A_0(t_1-\tau)} (B_0 m_1 + K_0 f(\tau)) d\tau + \right. \\ &\quad \left. + e^{A_0 t_1} \int_{t_1}^{T_B} e^{A_0(T_B-\tau)} (B_0 m_2 + K_0 f(\tau)) d\tau \right). \end{aligned} \quad (9)$$

If we look for the periodic solution of the system (1), (2) with an a priori given period, say, $T_B = kT$ with $k \in \mathbf{N}$ and $T = 2\pi/\omega$, then the system of the transcendental equations (8) depends on one variable t_1 only; and the system is analytically solvable with respect to the variable.

The analytical solution of the system (8), the parametrical conditions for its existence formulated as a theorem with the proof, and also an example confirming consistency of the conditions are given in [10].

The switching points X^1 and X^2 are uniquely determined by the analytical formulae (9). These formulae are obtained from the system of the transcendental equations composed according to the prescribed above sequence of motions for the image point of the periodic solution from one hyperplane to other one. It follows that the received switching points belong to the trajectory of the desired solution as well as to the hyperplanes. The existence of the unique periodic solution of the initial system with the given properties is proved. As it was shown above, the initial and the canonical systems are equivalent due to the nonsingular transformation. Hence the obtained results are true for the initial system as well. \square

Remark 1. If Condition 5) of Theorem 1 is not fulfilled, i.e. the system of the transcendental equations (8) has no solution t_1 for the given $k \in \mathbf{N}$, then it is possible to approve that the initial system has no periodic solutions with two switching points lying on the hyperplanes of the form $(C, Y) = \ell_i$ ($i = 1, 2$).

Remark 2. A.V. Pokrovskii [9] obtained powerful analytical results for systems of the considered class. He proved the existence of at least one asymptotically stable solution with the period equal to the period of external influence. In [9], the positivity condition of the system (i.e. restrictions on the coefficients of the feedback vector C) is stipulated and the matrix A is supposed to be hurwitzean. Comparing Theorem 1 with results in [9], we obtain Theorem 2.

Theorem 2. *Let the following conditions hold:*

1) *the external influence $f(t)$ of the system (1) is a T -periodic function of the form (2), where $T = 2\pi/\omega$;*

2) *the system (1) is reduced to the canonical form (3) by the nonsingular transformation $Y = SX$, where the form of the matrix S is determined by the formula (5), under the following conditions:*

a) *the matrix A has the prime, nonzero, and real eigenvalues λ_i ($i = \overline{1, n}$) such that $\lambda_j < 0$ ($j = \overline{1, n}$, $j \neq s$);*

b) *$\det \|B, AB, A^2 B, \dots, A^{n-1} B\| \neq 0$;*

3) *$\sum_{k=1}^n c_k N_k(\lambda_j) = 0$ ($j = \overline{1, n}$, $j \neq s$);*

4) *(t_1, T, X^1, X^2) is the solution to the system of the transcendental equations (8), (9) with the parameters satisfying to its solvability conditions for $\gamma_s > 0$, $m_1 \geq 0$.*

Then the system (1) has the unique asymptotically stable T -periodic solution with two switching points belonging to the hyperplanes of the form $\sigma = \ell_i$, where $i = 1, 2$.

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Существование единственного kT -периодического решения для одного класса нелинейных систем

Виктория В. Евстафьева

Рассматривается система обыкновенных дифференциальных уравнений с разрывной гистерезисной нелинейностью и внешним непрерывным бигармоническим возмущением. Получены достаточные условия существования периодических решений с заданными свойствами, в частности, с периодом, равным и кратным периоду внешнего воздействия. Использован подход к выбору коэффициентов обратной связи, основывающийся на неособом преобразовании исходной системы для получения канонической системы специального вида и позволяющий в аналитическом виде находить моменты времени и точки переключения изображающей точки искомого решения.

Ключевые слова: вынужденные периодические колебания, неособые преобразования, управляемые системы, разрывная гистерезисная нелинейность, моменты времени и точки переключения.